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A NEW PROOF OF A THEOREM OF JAYNE AND ROGERS

Abstract

We give a new simpler proof of a theorem of Jayne and Rogers.

1   Introduction

In this paper we will give a new proof of a Jayne–Rogers theorem. First recall from [4] the following definitions:

Definition 1. Let X, Y be metric spaces. A function $f \colon X \to Y$ is said to be $\mathbf{\Delta}_2^0$-*function* if $f^{-1}(S) \in \mathbf{\Sigma}_2^0$ for every $S \in \mathbf{\Sigma}_2^0$ (equivalently, $f^{-1}(U) \in \mathbf{\Delta}_2^0$ for every open $U \subseteq Y$). Sometimes these functions are also called *first level Borel functions* (see [4]).

The function f is said to be *piecewise continuous* if X can be expressed as the union of an increasing sequence $X_0, X_1, \ldots$ of closed sets such that $f \restriction X_n$ is continuous for every $n \in \omega$.

Obviously, if $f \colon X \to Y$ is piecewise continuous and $X' \subseteq X$ then $f \restriction X'$ is piecewise continuous as well. Observe also that f is piecewise continuous if and only if there is a $\mathbf{\Delta}_2^0$-partition $\langle D_n \mid n \in \omega \rangle$ of X such that $f \restriction D_n$ is continuous for every $n \in \omega$. For one direction, if f is piecewise continuous then putting $D_0 = X_0$ and $D_{n+1} = X_{n+1} \setminus X_n$ we have the desired partition. Conversely, let $P_{m,n} \in \mathbf{\Pi}_1^0$ be such that $D_n = \bigcup_{m \in \omega} P_{m,n}$ and $P_{m,n} \subseteq P_{m',n}$ for every $m \leq m'$ and $n \in \omega$, and let $X_n = \bigcup_{i \leq n} P_{n,i}$. It is easy to check

that the X_n are increasing and closed, and that $f \upharpoonright X_n$ is continuous (since $P_{n,i} \cap P_{n,j} = \emptyset$ whenever $i \neq j$). In the rest of this paper, when we will refer to some piecewise continuous function we will generally have in mind a function with this "partition" property. Finally, a third equivalent and useful definition is that X can be covered by a (not necessarily increasing) countable family $P_0, P_1, \ldots$ of closed sets such that $f \upharpoonright P_n$ is continuous for every $n \in \omega$.

Definition 2. A set S in a metric space is said to be Souslin-$\mathscr{F}$ set if it belongs to $\mathcal{A}\mathbf{\Pi}_1^0$, where $\mathcal{A}$ is the usual Souslin operation (see [5, Definition 25.4]).

A metric space X is said to be an *absolute Souslin-$\mathscr{F}$ set* if X is a Souslin-$\mathscr{F}$ set in the completion of X under its metric.

Observe that if X is separable then it is an absolute Souslin-$\mathscr{F}$ set if and only if it is Souslin, that is if and only if it is the continuous image of the Baire space $^\omega\omega$.

Now we are ready to give the statement of the Jayne–Rogers theorem.

Theorem 1.1 (Jayne–Rogers)**.** *If X is an absolute Souslin-$\mathscr{F}$ set, then $f \colon X \to Y$ is a $\mathbf{\Delta}_2^0$-function if and only if it is piecewise continuous.*

According to the authors of [4], their proof "even in the case when X and Y are separable, is complicated". Sixteen years later, Sławomir Solecki provided in [6] a new proof of Theorem 1.1 in the case when X and Y are separable and X is Souslin (in fact he proved a much stronger result which refines Theorem 1.1), but even in that case the proof was quite complicated. Our goal is to provide a simpler proof of Theorem 1.1. Our proof is divided into two steps: first we will prove the nontrivial direction of Theorem 1.1 with the auxiliary assumptions that X is completely metrizable and f is of Baire class 1 (see Theorem 2.1), and then we will use a combination of several well-known results to prove Theorem 1.1 as a corollary of Theorem 2.1. The authors would like to thank the anonymous referee for suggesting a way for removing the condition of separability on the spaces involved.

We will assume $\mathsf{ZF} + \mathsf{DC}(\mathbb{R})$ throughout the paper (note that the Jayne–Rogers' and Solecki's proofs are carried out in ZFC, but by a simple absoluteness argument the result must hold also in $\mathsf{ZF} + \mathsf{DC}(\mathbb{R})$). *All spaces considered are metric.* Our notation will be quite standard: the set of the natural numbers will be denoted by ω, while if X is any topological space and A is a subset of X we will denote the closure of A with $\mathrm{Cl}(A)$. The set of all binary sequences of finite length will be denoted by $^{<\omega}2$, and $^\omega 2$ will denote the Cantor space. A function $f \colon X \to Y$ will be said of *Baire class* 1 if it is the pointwise limit of a sequence of continuous functions $f_n \colon X \to Y$. Finally, if (X, d) is any

metric space, a set $U \subseteq X$ will be called *basic open* if it is an open ball of X, i.e. if $U = \{x \in X \mid d(x, x_0) < r\}$ where $x_0 \in X$ and $r \in \mathbb{R}^+$. For all the other undefined symbols and notions we refer the reader to the standard monograph [5].

2   The proof of the Jayne–Rogers theorem

The main result of this paper is the following theorem, from which the Jayne–Rogers theorem will follow.

Theorem 2.1. *Let X and Y be metric spaces such that the metric of X is complete, and let $f\colon X \to Y$ be of Baire class 1. If f is a $\mathbf{\Delta}_2^0$-function then it is piecewise continuous.*

Recall that if $f\colon X \to Y$ is of Baire class 1 then it is also $\mathbf{\Sigma}_2^0$-*measurable*, i.e. $f^{-1}(U) \in \mathbf{\Sigma}_2^0$ for every open set $U \subseteq Y$, but the converse in general fails. Nevertheless, if we require that X is a zero-dimensional absolute Souslin-$\mathscr{F}$ set then f is of Baire class 1 just in case it is $\mathbf{\Sigma}_2^0$-measurable (see [2, Theorem 8]). Recall also that a family $\mathcal{B}$ of subsets of X is said to be *discrete* if X can be covered by open sets each having a nonvoid intersection with at most one member of $\mathcal{B}$ (in particular, the elements of $\mathcal{B}$ must be pairwise disjoint). If $\mathcal{B}$ is a discrete family, then the following facts easily follow from the definition:

- $\mathrm{Cl}(\mathcal{B}) = \{\mathrm{Cl}(B) \mid B \in \mathcal{B}\}$ is discrete;

- if $\mathcal{B}'$ is a family of subsets of X and there is an injection $j\colon \mathcal{B}' \to \mathcal{B}$ such that $B' \subseteq j(B')$ for every $B' \in \mathcal{B}'$ (e.g. if $\mathcal{B}' \subseteq \mathcal{B}$), then $\mathcal{B}'$ is discrete as well;

- if each $B \in \mathcal{B}$ is closed then $\bigcup \mathcal{B}$ is also closed;

- if $f\colon X \to Y$ is such that $f \upharpoonright B$ is continuous for every $B \in \mathcal{B}$, then $f \upharpoonright \bigcup \mathcal{B}$ is continuous.

The following construction will be used a couple of times: let g be any function defined on a metric space Z, let $\mathcal{F}_g$ be the collection of all closed sets C of the completion of Z such that $g \upharpoonright (C \cap Z)$ is continuous, and let $\mathcal{I}_g$ be the σ-ideal of the subsets of the fixed completion of Z that can be covered by countably many elements of $\mathcal{F}_g$ (note in particular that $A \subseteq Z$ belongs to $\mathcal{I}_g$ if and only if $g \upharpoonright A$ is piecewise continuous).

Lemma 2.2. *$\mathcal{I}_g$ is closed under discrete unions.*

Proof. Let $\mathcal{B}$ a discrete family of subsets of Z and assume $\mathcal{B} \subseteq \mathcal{I}_g$. Let F_n^B (for $B \in \mathcal{B}$ and $n \in \omega$) be closed sets such that $B \subseteq \bigcup_n F_n^B$ and $g \upharpoonright F_n^B$ is continuous. We can assume without loss of generality that $F_n^B \subseteq \mathrm{Cl}(B)$ (if not simply replace F_n^B by $F_n^B \cap \mathrm{Cl}(B)$). Put $\mathcal{F}_n = \{F_n^B \mid B \in \mathcal{B}\}$. Since the function $j \colon \mathcal{F}_n \to \mathrm{Cl}(\mathcal{B})$ which maps F_n^B to $\mathrm{Cl}(B)$ is injective, by the facts aboute discrete families mentioned above we get that $\mathrm{Cl}(\mathcal{B})$, and hence also each $\mathcal{F}_n$, must be discrete: but this implies that $F_n = \bigcup \mathcal{F}_n$ is closed and $g \upharpoonright F_n$ is continuous. Therefore $\bigcup \mathcal{B} \in \mathcal{I}_g$ because $\bigcup \mathcal{B} \subseteq \bigcup_{B \in \mathcal{B}} \left(\bigcup_n F_n^B \right) = \bigcup_n \left(\bigcup_{B \in \mathcal{B}} F_n^B \right) = \bigcup_n F_n$. $\qquad\square$

Proof of Theorem 1.1. One direction is trivial. For the other direction, assume toward a contradiction that f is a $\boldsymbol{\Delta}_2^0$-function but *not* piecewise continuous. Let $\mathcal{I} = \mathcal{I}_f$ be defined as before (with $Z = X$). By [3, Proposition 3.5], Lemma 2.2 implies that $\mathcal{I}$ is locally determined, and since it is trivially $\boldsymbol{F}_\sigma$ supported we can apply [3, Theorem 1.3]: therefore, either $X \in \mathcal{I}$ or there is $\tilde{X} \subseteq X$ such that $\tilde{X}$ is a $\boldsymbol{\Pi}_2^0$-subset of the completion of X (hence a completely metrizable space) and $\tilde{X} \notin \mathcal{I}$. Moreover, inspecting the proof of [3, Theorem 1.3] it is easy to check that the $\tilde{X}$ obtained in the second case is also zero-dimensional. Since the first alternative easily implies that f is piecewise continuous, we can assume that the second alternative holds and therefore that $f' = f \upharpoonright \tilde{X}$ is not piecewise continuous. Note that we can assume also that f' is of Baire class 1 (otherwise, by [2, Theorem 8] we would have that f' is not even $\boldsymbol{\Sigma}_2^0$-measurable and hence not a $\boldsymbol{\Delta}_2^0$-function), and therefore we can apply Theorem 2.1 to f'; this gives the desired contradiction. $\qquad\square$

The strategy for the proof of Theorem 2.1 will be as follows: we will assume that $f \colon X \to Y$ is of Baire class 1 (hence also $\boldsymbol{\Sigma}_2^0$-measurable) but not piecewise continuous, and then we will prove that f can not be a $\boldsymbol{\Delta}_2^0$-function by constructing an open set $\hat{U} \subseteq Y$ such that $f^{-1}(\hat{U})$ is a $\boldsymbol{\Sigma}_2^0$-complete set. To prove that $f^{-1}(\hat{U})$ is $\boldsymbol{\Sigma}_2^0$-complete, we will construct (together with $\hat{U}$) a continuous reduction from the well-known $\boldsymbol{\Sigma}_2^0$-complete set

$$S = \{z \in {}^\omega 2 \mid z(n) \text{ eventually equals } 0\} = \{z \in {}^\omega 2 \mid \exists i \forall j \geq i (z(j) = 0)\}$$

to $f^{-1}(\hat{U})$, i.e. a continuous function $g \colon {}^\omega 2 \to X$ such that for all $z \in {}^\omega 2$

$$z \in S \iff g(z) \in f^{-1}(\hat{U}).$$

The construction of $\hat{U}$ and g will be carried out by inductively localizing the property of not being piecewise continuous of f to smaller and smaller subsets of X. However, before proving Theorem 2.1 we need a couple of technical lemmas. For the next few results, X' *will be an arbitrary subset of* X. Given $A, B \subseteq Y$ we will say that A and B are *strongly disjoint* if $\mathrm{Cl}(A) \cap \mathrm{Cl}(B) = \emptyset$.

Moreover if $h\colon X' \to Y$ is any function we put $A^h = h^{-1}(Y \setminus \mathrm{Cl}(A))$. Note that for every $A, B \subseteq Y$ one has $(A \cup B)^h = A^h \cap B^h$. If h is $\mathbf{\Sigma}_2^0$-measurable and $A, B \subseteq Y$ are strongly disjoint, then we have that if $h \restriction A^h$ and $h \restriction B^h$ are both piecewise continuous then the whole h is piecewise continuous. In fact, A^h and B^h is a finite $\mathbf{\Sigma}_2^0$-covering of X' (by the strongly disjointness of A and B), which by the reduction property of $\mathbf{\Sigma}_2^0$ can be refined to a $\mathbf{\Delta}_2^0$-partition $\langle D_0, D_1 \rangle$ of X' such that $D_0 \subseteq A^h$, $D_1 \subseteq B^h$, and hence both $h \restriction D_0$ and $h \restriction D_1$ are piecewise continuous. But if $h'\colon X' \to Y$ is such that for some $\mathbf{\Delta}_2^0$-partition $\langle D_n' \mid n \in \omega \rangle$ of X' we have that $h' \restriction D_n'$ is piecewise continuous for every n, then h' is piecewise continuous on the whole X': therefore h is piecewise continuous as well.

Now let $h\colon X' \to Y$ be a $\mathbf{\Sigma}_2^0$-measurable function, $x \in X'$, and A be any subset of Y. We say that x is h-*irreducible outside* A if for every open neighborhood $V \subseteq X'$ of x the function $h \restriction A^h \cap V$ is not piecewise continuous, otherwise we say that x is h-*reducible outside* A. In our proofs the set A will be often of the form $A = U_0 \cup \ldots \cup U_n$ with $U_0, \ldots, U_n$ a sequence of pairwise strongly disjoint open sets. Notice that if x is h-irreducible outside A then $x \in \mathrm{Cl}(A^h)$, as otherwise $A^h \cap V = \emptyset$ for some open neighborhood V of x and therefore $h \restriction A^h \cap V$ would be trivially (piecewise) continuous. Moreover, if there are x and A such that x is h-irreducible outside A then clearly h can not be piecewise continuous. Finally, it is easy to check that if x is h-irreducible outside A and $A' \subseteq A$ then x is also h-irreducible outside A', and that if $X'' \subseteq X'$ and $x \in X''$ is h'-irreducible outside A (where $h' = h \restriction X''$) then x is also h-irreducible outside A.

Lemma 2.3. *Suppose $h\colon X' \to Y$ is a $\mathbf{\Sigma}_2^0$-measurable function and $U_0, \ldots, U_n \subseteq Y$ are basic open sets of Y such that $\mathrm{range}(h) \cap \mathrm{Cl}(U_i) = \emptyset$ for every $i \leq n$. Then h is* not *piecewise continuous if and only if* (∗) *there is an $x \in X'$ and a basic open set $U \subseteq Y$ strongly disjoint from $U_0, \ldots, U_n$ such that $h(x) \in U$ and x is h-irreducible outside U.*

Proof. Put $C = \mathrm{Cl}(U_0) \cup \ldots \cup \mathrm{Cl}(U_n)$. We will prove that h is piecewise continuous if and only if (∗) does not hold. If h is piecewise continuous then the same must hold for $h \restriction X''$ where X'' is any subset of X', therefore one direction is trivial. For the other direction, assume toward a contradiction that (∗) does not hold, i.e. for every $x \in X'$ and every open set $U \subseteq Y$ strongly disjoint from C such that $h(x) \in U$ we have that x is h-reducible outside U, that is there is some open neighborhood $V \subseteq X'$ of x such that $h \restriction U^h \cap V$ is piecewise continuous. Since X is a metric space, and hence also paracompact, let $\mathcal{B} = \bigcup_n \mathcal{B}_n$ be a base for the topology of X such that each $\mathcal{B}_n$ is discrete (see [1]). Then let Q_n be the union of the elements of $\mathcal{B}_n$ which belongs to $\mathcal{I} = \mathcal{I}_h$, so that each Q_n belongs to $\mathcal{I}$ by Lemma 2.2. Finally put $Q = \bigcup_n Q_n$ and

notice that $h \restriction Q$ is piecewise continuous since $Q \in \mathcal{I}$, and that Q is open and contains as a subset each open set W for which $h \restriction W$ is piecewise continuous. We claim that $h \restriction X' \setminus Q$ is continuous (from this easily follows that h is piecewise continuous). Suppose otherwise, so that given any $x \in X' \setminus Q$ and any open set $U \subseteq Y$ such that $h(x) \in U$ there is no open neighborhood V of x such that $h(V \cap (X' \setminus Q)) \subseteq U$. Fix such an x and U, and let $U' \subseteq Y$ be basic open, strongly disjoint from C, and such that $h(x) \in U'$ and $\mathrm{Cl}(U') \subseteq U$ (U' exists since Y is metric). Let $V \subseteq X'$ be given by the failure of $(*)$ on the inputs x and U': by our hypothesis there is $x' \in V \cap (X' \setminus Q)$ such that $h(x') \notin \mathrm{Cl}(U')$, and clearly we can find a basic open $U'' \subseteq Y$ strongly disjoint from U' and C, and such that $h(x') \in U''$. Let $V' \subseteq X'$ be the open set given by the failure of $(*)$ on inputs x' and U''. Since V and V' have been chosen in such a way that $h \restriction (U')^h \cap V$ and $h \restriction (U'')^h \cap V'$ are piecewise continuous, and since $\{(U')^h \cap V, (U'')^h \cap V'\}$ is a $\mathbf{\Sigma}_2^0$-covering of $V \cap V'$, by the strong disjointness of U' and U'' we must have that $h \restriction V \cap V'$ is piecewise continuous, and therefore $V \cap V' \subseteq Q$: but this implies that $x' \in Q$, a contradiction! $\qquad\square$

Lemma 2.4. *Let* $h \colon X' \to Y$ *be a* $\mathbf{\Sigma}_2^0$-*measurable function,* $x \in X'$, $A \subseteq Y$, *and* $U_0, \ldots, U_n$ *be a sequence of pairwise strongly disjoint open subsets of* Y. *If* x *is* h-*irreducible outside* A *then there is at most one* $i \leq n$ *such that* x *is* h-*reducible outside* $A \cup U_i$.

Proof. Assume that $i \leq n$ is such that x is h-reducible outside $A \cup U_i$, i.e. that there is an open neighborhood $V \subseteq X'$ of x such that $h \restriction (A \cup U_i)^h \cap V$ is piecewise continuous. If there were some $j \neq i$ with the same property, then there must be some open neighborhood $W \subseteq X'$ of x such that $h \restriction (A \cup U_j)^h \cap W$ is piecewise continuous. But since U_i and U_j are strongly disjoint, this would imply that $h \restriction A^h \cap V \cap W$ is piecewise continuous as well, and thus $V \cap W$ would contradict the fact that x is h-irreducible outside A. $\qquad\square$

Finally observe that if $f \colon X \to Y$ is the pointwise limit of a sequence of functions $\langle f_m \colon X \to Y \mid m \in \omega \rangle$, then we have the following property: if $x \in X$ and $U_0, U_1, \ldots$ are pairwise disjoint open sets such that for infinitely many n's there is an m for which $f_m(x) \in U_n$, then $f(x) \notin U_n$ for each n (otherwise, $f_m(x) \in U_n$ for all but finitely many m's contradicting our hypothesis).

Now we are ready to prove Theorem 2.1. The proof essentially uses recursively Lemma 2.3 applied to smaller and smaller subspaces of X to construct some sequences, and Lemma 2.4 will guarantee that at each stage the construction can be carried out. This is the reason for which we have proved both the lemmas for arbitrary functions h with domain an arbitrary subset X' of X: in fact we will generally apply them to the restriction of the original function f to some subset of X, that is with $h = f \restriction X'$.

Proof of Theorem 2.1. Assume that $f\colon X \to Y$ is of Baire class 1 (hence also $\mathbf{\Sigma}_2^0$-measurable) but not piecewise continuous, and let $\langle f_n \mid n \in \omega \rangle$ be a sequence of continuous functions which pointwise converges to f. As explained on page 4, we will inductively construct an open set $\hat{U} \subseteq Y$ and a continuous reduction $g\colon {}^\omega 2 \to X$ from $S = \{z \in {}^\omega 2 \mid \exists i \forall j \geq i(z(j) = 0)\}$ to $f^{-1}(\hat{U})$. The function g will be defined using a *weak* Cantor scheme $\langle V_s \mid s \in {}^{<\omega}2 \rangle$ (that is a classical Cantor scheme in which we drop the condition $V_{s \smallfrown 0} \cap V_{s \smallfrown 1} = \emptyset$) such that for every $s, t \in {}^{<\omega}2$ we have:

1) V_s is an open subset of X;

2) if $s \subsetneq t$ then $\mathrm{Cl}(V_t) \subseteq V_s$;

3) $\mathrm{diam}(V_s) \leq 2^{-\mathrm{length}(s)}$.

It is straightforward to check that, given such a scheme, the function $g\colon {}^\omega 2 \to X\colon z \mapsto \bigcap_{n \in \omega} V_{z \restriction n}$ is well-defined (by the completeness of X) and continuous (in fact it is Lipschitz with constant 1).

The construction will be carried out by recursion on the rank of $s \in {}^{<\omega}2$ with respect to the order $\preceq$ defined by

$$s \preceq t \iff \mathrm{length}(s) < \mathrm{length}(t) \vee (\mathrm{length}(s) = \mathrm{length}(t) \wedge s \leq_{\mathrm{lex}} t),$$

where $\leq_{\mathrm{lex}}$ is the usual lexicographical order on ${}^{<\omega}2$ (the strict part of $\preceq$ will be denoted by $\prec$). In fact we will define, together with a scheme $\langle V_s \mid s \in {}^{<\omega}2 \rangle$ with the properties above, a sequence $\langle x_s \mid s \in {}^{<\omega}2 \rangle$ of points of X and a sequence $\langle U_s \mid s \in {}^{<\omega}2 \rangle$ of subsets of Y such that for every $s \in {}^{<\omega}2$:

  i) $x_s \in V_s$;

  ii) $f(x_s) \in U_s$;

  iii) U_s is basic open and for every $t \in {}^{<\omega}2$ we have that U_s and U_t are either equal or strongly disjoint;

  iv) there is some $m \in \omega$ such that $f_m(V_s) \subseteq U_s$;

  v) x_t is f-irreducible outside A for every $t \preceq s$, where $A = \bigcup_{u \preceq s} U_u$;

  vi) if the last digit of s is 1 then $U_s \neq U_t$ for every $t \prec s$ (and therefore, in particular, for every $t \subsetneq s$).

As already noted, to construct these sequences we will recursively apply Lemma 2.3 to the restriction of f to smaller and smaller pieces.

At the first stage, let x and U be given as in Lemma 2.3 applied to the whole f, and let $V = f_m^{-1}(U)$ where $m \in \omega$ is such that $f_m(x) \in U$ (such an

m must exists by the fact that f is the limit of the f_n's). Then put $V_\emptyset = V$, $x_\emptyset = x$ and $U_\emptyset = U$. Now let $s \neq \emptyset$ and suppose we have defined V_t, x_t and U_t for $t \prec s$. Put $s^- = s \restriction (\text{length}(s) - 1)$. If the last digit of s is a 0, then simply put $V_s = W$, $x_s = x_{s^-}$ and $U_s = U_{s^-}$, where W is any open set such that $\text{Cl}(W) \subseteq V_{s^-}$, $x_s \in W$ and $\text{diam}(W) \leq 2^{-\text{length}(s)}$. Otherwise the last digit of s is 1: by the inductive hypothesis, condition v) implies that $h_0 = f \restriction A^f \cap V_{s^-}$, where $A = \bigcup_{t \prec s} U_t$, is not piecewise continuous (otherwise, since $x_{s^- \frown 0} \in V_{s^- \frown 0} \subseteq V_{s^-}$, $x_{s^- \frown 0}$ should be f-reducible outside A).

Claim. There are $x_s \in V_{s^-}$ and $U_s \subseteq Y$ such that $f(x_s) \in U_s$, U_s is basic open and strongly disjoint from A (which in particular implies $U_s \neq U_t$ for every $t \prec s$), and x_t is f-irreducible outside $A \cup U_s$ for every $t \preceq s$.

Proof of the Claim. Let $k = |\{t \in {}^{<\omega}2 \mid t \prec s\}|$. Using Lemma 2.3, for $j \leq k+1$ recursively construct x_j and U_j such that each x_j belongs to V_{s^-}, $f(x_j) \in U_j$, U_j is strongly disjoint from $A \cup U_{<j}$ (where $U_{<j} = \emptyset$ if $j = 0$ and $U_{<j} = \bigcup_{i<j} U_i$ otherwise), and x_j is h_j-irreducible outside $A \cup U_{<j} \cup U_j$ (hence in particular x_j is f-irreducible outside $A \cup U_j$), where h_0 is as before and $h_{j+1} = h_j \restriction (A \cup U_{<(j+1)})^f$. Now notice that there must be $\bar{j} \leq k+1$ such that the claim is satisfied with $x_s = x_{\bar{j}}$ and $U_s = U_{\bar{j}}$: if not, by the pigeonhole principle there should be $j \neq j' \leq k+1$ and $t \prec s$ such that x_t is f-reducible both outside $A \cup U_j$ and $A \cup U_{j'}$, contradicting Lemma 2.4. $\square$ *Claim*

Let $W \subseteq X$ be an open neighborhood of x_s such that $\text{diam}(W) \leq 2^{-\text{length}(s)}$, $\text{Cl}(W) \subseteq V_{s^-}$ and $f_m(W) \subseteq U_s$ for some m, and define $V_s = W$. This completes the recursive definition of the sequences required.

It is easy to check that the scheme $\langle V_s \mid s \in {}^{<\omega}2 \rangle$ and the sequences $\langle x_s \mid s \in {}^{<\omega}2 \rangle$ and $\langle U_s \mid s \in {}^{<\omega}2 \rangle$ constructed in this way are as required, i.e. that they satisfy 1)–3) and i)–vi). Now put $\hat{U} = \bigcup_{s \in {}^{<\omega}2} U_s$, and let $g \colon {}^{\omega}2 \to X$ be obtained from $\langle V_s \mid s \in {}^{<\omega}2 \rangle$ as described above. We have only to check that g is a reduction of S to $f^{-1}(\hat{U})$. Let $\langle U_k \mid k \in \omega \rangle$ be an enumeration without repetitions of $\langle U_s \mid s \in {}^{<\omega}2 \rangle$, so that by condition iii) the U_k's are pairwise disjoint and $\hat{U} = \bigcup_{k \in \omega} U_k$. If $z \in S$, then for some $\bar{n} \in \omega$ we will have that $x_{z \restriction m} = x_{z \restriction \bar{n}} = \bar{x}$ for every $m \geq \bar{n}$, therefore $g(z) = \bar{x}$ and $f(g(z)) = f(\bar{x}) \in U_{z \restriction \bar{n}} \subseteq \hat{U}$. Assume now $z \notin S$: by conditions vi) and iv), for infinitely many k's there is some $m \in \omega$ such that $f_m(g(z)) \in U_k$ (since $g(z) \in V_{z \restriction n}$ for every $n \in \omega$), and therefore $f(g(z)) \notin \hat{U}$ by the observation preceding this proof. $\square$

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